

# FREE POISSON HOPF ALGEBRAS GENERATED BY COALGEBRAS

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**ABSTRACT.** We construct the analogue of Takeuchi's free Hopf algebra in the setting of Poisson Hopf algebras. More precisely, we prove that there exists a free Poisson Hopf algebra on any coalgebra or, equivalently that the forgetful functor from the category of Poisson Hopf algebras to the category of coalgebras has a left adjoint. In particular, we also prove that the category of Poisson Hopf algebras is a reflective subcategory of the category of Poisson bialgebras. Along the way, we describe coproducts and coequalizers in the category of Poisson Hopf algebras, therefore showing that the latter category is cocomplete.

## INTRODUCTION

A Poisson Hopf algebra is both a Poisson algebra and a Hopf algebra such that the comultiplication and the counit are morphisms of Poisson algebras. Such objects are situated at the border between Poisson geometry [9] and quantum groups [6]. Poisson structures are known for making striking and unexpected appearances in a variety of different fields of mathematics or mathematical physics such as differential geometry, both classical and quantum mechanics, Lie groups and representation theory, algebraic geometry, etc. Furthermore, as it turns out, many important objects carry not only a Poisson structure but also a natural Poisson Hopf algebra structure; we only mention here the algebra of smooth functions on a Poisson group or the polynomial algebra  $\mathbb{C}[\mathfrak{g}^*]$  of a finite dimensional complex simple Lie algebra  $\mathfrak{g}$ , see e.g. [6, 7, 9, 22] for more details and further examples. On the other hand, there are also examples of Hopf algebras carrying an induced Poisson structure which makes it into a Poisson Hopf algebra. For instance, the graded algebra associated to any connected Hopf algebra is such an example [24]. It is therefore important to achieve a good understanding of these structures as well as to be able to construct new ones. The present paper is a contribution to the study of Poisson Hopf algebras from an algebraic point of view. More precisely, in light of the increasing interest shown recently in the category of Hopf algebras (see [2, 3, 4, 5, 8, 17, 18, 19, 23] and the references therein), we will study the categorical properties of Poisson Hopf algebras. As we will see, the category  $k\text{-PoissBiAlg}$  of Poisson bialgebras (and therefore the category of Poisson Hopf algebras) is not as friendly as the category  $k\text{-BiAlg}$  of

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bialgebras in the sense that it does not enjoy the same nice symmetry. More precisely, it is well-known that for any field  $k$  the category  $k\text{-BiAlg}$  of bialgebras is isomorphic to  $\mathbf{Mon}(\mathbf{Comon}({}_k\mathcal{M}))$  as well as to  $\mathbf{Comon}(\mathbf{Mon}({}_k\mathcal{M}))$  while the category  $k\text{-PoissBiAlg}$  is only isomorphic to  $\mathbf{Comon}(k\text{-Poiss})$ , where  $\mathbf{Mon}(\mathcal{C})$  and  $\mathbf{Comon}(\mathcal{C})$  denote the monoids, respectively the comonoids, of a given category  $\mathcal{C}$  (see for instance [19]) and  $k\text{-Poiss}$  stands for the category of Poisson algebras. Therefore, the main drawback in studying Poisson Hopf algebras is that the duality arguments usually employed for Hopf algebras do not hold anymore.

An outline of the paper is as follows. In Section 1 we introduce the notation and recall briefly the basic concepts needed in the sequel. Section 2 contains the explicit description of coproducts and coequalizers in the categories of Poisson algebras, Poisson bialgebras as well as Poisson Hopf algebras. As we will see, all these constructions rely on those performed in the category of Poisson algebras. The main results of the paper, namely the explicit constructions of the free Poisson Hopf algebra generated by a coalgebra, respectively a Poisson bialgebra, are proved in Section 3. The constructions in this section parallel those in [21]. However, it is worth pointing out that the construction of the free Poisson Hopf algebra on a coalgebra does not follow trivially from Takeuchi's construction. What we mean, precisely, is that we do not merely put a Lie algebra structure on Takeuchi's free Hopf algebra, the free Poisson Hopf algebra on a Poisson bialgebra being obtained by a different construction. The paper ends with some open problems concerning the existence of limits and cofree objects in the above mentioned categories. Another important issue, also connected to the existence of cofree objects, which still needs to be addressed is the injectivity (resp. surjectivity) of monomorphisms (resp. epimorphisms) in the category of Poisson Hopf algebras.

## 1. PRELIMINARIES

Throughout this paper,  $k$  will be a field. Unless specified otherwise, all vector spaces, tensor products, homomorphisms, algebras, coalgebras, bialgebras, Lie algebras, Poisson algebras, Hopf algebras and Poisson Hopf algebras are over  $k$  and all algebras are considered to be unital. Our notation for the standard categories is as follows:  ${}_k\mathcal{M}$  ( $k$ -vector spaces),  $k\text{-Alg}$  (associative unital  $k$ -algebras),  $k\text{-Lie}$  (Lie algebras over  $k$ ),  $k\text{-Poiss}$  (Poisson algebras over  $k$ ),  $k\text{-BiAlg}$  (bialgebras over  $k$ ),  $k\text{-HopfAlg}$  (Hopf algebras over  $k$ ),  $k\text{-BiAlgPoiss}$  (Poisson bialgebras over  $k$ ),  $k\text{-HopfPoiss}$  (Poisson Hopf algebras over  $k$ ). For a coalgebra  $C$ , we will use Sweedler's  $\Sigma$ -notation  $\Delta(c) = c_{(1)} \otimes c_{(2)}$  with suppressed summation sign.  $C^{\text{cop}}$  stands for the coopposite of the coalgebra  $C$ . For a coalgebra  $C$  and an algebra  $A$ ,  $\text{Hom}_k(C, A)$  becomes an algebra with respect to the convolution product  $*$ ; more precisely, we have  $(f * g)(c) = f(c_{(1)})g(c_{(2)})$ . If  $A$  is an algebra then the vector space  $A$  together with the product  $[-, -] : A \times A \rightarrow A$  defined by  $[a, b] = ab - ba$  is a Lie algebra denoted by  $A^-$ . Given a vector space  $V$ ,  $(T(V), i)$  stands for the *tensor algebra* on  $V$ , where  $i : V \rightarrow T(V)$  is the canonical inclusion. The Lie subalgebra of  $T(V)^-$  generated by  $i(V)$  is called the *free Lie algebra* generated by  $V$  and will be denoted by  $L(V)$ . The *symmetric algebra* on  $V$  will be denoted by  $(S(V), i)$ . If  $\mathfrak{g}$  is a Lie algebra with bracket  $[\cdot, \cdot]$  then the symmetric algebra  $S(\mathfrak{g})$  inherits a Poisson algebra

structure  $\{\cdot, \cdot\}$  in a canonical way, namely  $\{g, h\} = [g, h]$  for all  $g, h \in \mathfrak{g}$ . Furthermore, we denote by  $(\mathcal{P}(V), \bar{i})$  the *free Poisson algebra* on  $V$ , where  $\mathcal{P}(V) = S(L(V))$  and  $\bar{i} : V \rightarrow \mathcal{P}(V)$  is the canonical inclusion; see [13] and the references therein for more details on the free Poisson algebra. As the terminology suggests, the functor sending a vector space  $V$  to the free Poisson algebra  $\mathcal{P}(V)$  provides a left adjoint to the forgetful functor  $U : k\text{-Poiss} \rightarrow {}_k\mathcal{M}$ .

Let us recall briefly some well known results pertaining to category theory, referring the reader to [1, 12] for more details. A category  $\mathcal{C}$  is called *(co)complete* if all diagrams in  $\mathcal{C}$  have (co)limits in  $\mathcal{C}$ . A category  $\mathcal{C}$  is *(co)complete* if and only if  $\mathcal{C}$  has (co)equalizers of all pairs of arrows and all (co)products [15, Theorem 6.10]. For a more detailed discussion concerning the completeness and cocompleteness of some of the categories mentioned above we refer the reader to [2, 3, 17, 18]. A full subcategory  $\mathcal{D}$  of  $\mathcal{C}$  is called *(co)reflective* in  $\mathcal{C}$  when the inclusion functor  $U : \mathcal{D} \rightarrow \mathcal{C}$  has a (right) left adjoint. A very convenient way of proving, in a constructive way, that a given covariant functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  has a left adjoint is by showing that for any object  $X \in \mathcal{D}$  the co-universal problem generated by  $X$  and  $F$  has a co-universal solution. More precisely, given  $X \in \mathcal{D}$ , a co-universal solution for the co-universal problem generated by  $X$  and  $F$  consists of an object  $G(X) \in \mathcal{C}$  and a map  $i : X \rightarrow F(G(X))$  in  $\mathcal{D}$  such that for each  $Y \in \mathcal{C}$  and for each map  $f : X \rightarrow F(Y)$  in  $\mathcal{D}$  there is a unique map  $g : G(X) \rightarrow Y$  in  $\mathcal{C}$  such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{i} & F(G(X)) \\ & \searrow f & \downarrow F(g) \\ & & F(Y) \end{array}$$

If for each  $X \in \mathcal{D}$  the co-universal problem defined by  $X$  and  $F$  has a co-universal solution, then  $G : \mathcal{D} \rightarrow \mathcal{C}$  defines a functor which is a left adjoint to  $F$  [14].

Recall that a *Poisson algebra* is both an associative commutative algebra and a Lie algebra living on the same vector space  $P$  such that any hamiltonian  $[p, -] : P \rightarrow P$  is a derivation of the associative algebra  $P$ , i.e. for all  $p, q, r \in P$  we have:

$$[p, qr] = [p, q]r + q[p, r]$$

If  $P_1, P_2$  are Poisson algebras then  $P_1 \otimes P_2$  has a Poisson algebra structure defined for all  $p, r \in P_1$  and  $q, s \in P_2$  by:

$$(1) \quad (p \otimes q) \cdot (r \otimes s) := pr \otimes qs, \quad [p \otimes q, r \otimes s]_{P \otimes P} := pr \otimes [q, s] + [p, r] \otimes qs$$

A linear map  $f : P_1 \rightarrow P_2$  is called a *morphism of Poisson algebras* if  $f$  is both an algebra map as well as a Lie algebra map. Furthermore, the category  $k\text{-Poiss}$  of Poisson algebras is in fact a monoidal category with the tensor product defined above. A *Poisson ideal* is a linear subspace which is both an ideal with respect to the associative product as well as a Lie ideal. If  $\mathcal{I}$  is a Poisson ideal of  $P$  then  $P/\mathcal{I}$  inherits a Poisson algebra structure in the obvious way.

A commutative bialgebra  $B$  together with a Poisson bracket  $[\cdot, \cdot]_B$  is called a *Poisson bialgebra* if the comultiplication  $\Delta_B$  and the counit  $\varepsilon_B$  are Poisson algebra maps, i.e.

besides from being algebra maps, for all  $a, b \in B$  we also have:

$$(2) \quad \Delta_B([a, b]) = [\Delta_B(a), \Delta_B(b)]_{B \otimes B}, \quad \varepsilon_B([a, b]) = [\varepsilon_B(a), \varepsilon_B(b)]_k$$

Let us observe that the second compatibility in (2) is trivially fulfilled as for all  $a, b \in B$  we have  $\varepsilon_B([a, b]) = 0$  (see [11]). Furthermore, if  $B$  is a Hopf algebra then  $B$  is called a *Poisson Hopf algebra*. It is straightforward to see that the antipode  $S_B$  is a Poisson algebra anti-morphism, i.e. for all  $a, b \in B$  we have:  $S_B([a, b]_B) = [S_B(b), S_B(a)]_B$ . A *morphism of Poisson bialgebras* is both a morphism of Poisson algebras and a morphism of coalgebras. A morphism of Poisson bialgebras between two Poisson Hopf algebras is automatically a Poisson Hopf morphism and therefore  $k\text{-HopfPoiss}$  is a full subcategory of the category  $k\text{-BiAlgPoiss}$ . It is straightforward to see that if  $(H, m, \eta, \Delta, \varepsilon, S, [-, -])$  is a Poisson Hopf algebra then  $(H^{\text{cop}}, m, \eta, \Delta, \varepsilon, S, [-, -]^{\text{op}})$ , where  $[a, b]^{\text{op}} = [b, a]$ , is again a Poisson Hopf algebra which we will denote by  $H^{\text{op}, \text{cop}}$ . We refer to [20] for further details concerning Hopf algebras and to [10] for a comprehensive treatment of Poisson algebras from both algebraic and geometrical point of view.

## 2. COLIMITS IN THE CATEGORY OF POISSON HOPF ALGEBRAS

The aim of this section is to construct coproducts and coequalizers in the category  $k\text{-PoissHopf}$  of Poisson Hopf algebras. Since there is a close connection between these constructions and the ones corresponding to the category  $k\text{-Poiss}$  of Poisson algebras we start by investigating the latter category. Our next result constructs coproducts and coequalizers in the category of Poisson algebras; as we will see, these constructions are obtained by modifying properly the objects which provide the coproducts respectively the coequalizers in the category  $k\text{-Alg}$  of algebras [14, page 50].

**Proposition 2.1.** *The category  $k\text{-Poiss}$  of Poisson algebras has arbitrary coproducts and coequalizers. In particular, the category  $k\text{-Poiss}$  of Poisson algebras is cocomplete.*

*Proof.* We first indicate the construction of coproducts. Let  $(P_l)_{l \in I}$  be a family of Poisson algebras and consider  $(\bigoplus_{l \in I} P_l, (j_l)_{l \in I})$  to be their coproduct in  $k\mathcal{M}$ . Then  $(\coprod_{l \in I} P_l := S(\bigoplus_{l \in I} P_l)/\overline{J}, \{-, -\}, (q_l)_{l \in I})$  is the coproduct of the above family in  $k\text{-Poiss}$ , where  $(S(\bigoplus_{l \in I} P_l), i)$  is the symmetric algebra on the vector space  $\bigoplus_{l \in I} P_l$ ,  $i : \bigoplus_{l \in I} P_l \rightarrow S(\bigoplus_{l \in I} P_l)$  stands for the canonical inclusion,  $\overline{J}$  is the Poisson ideal generated by the set  $J := \{i \circ j_l(x_l y_l) - i(j_l(x_l))i(j_l(y_l)), 1_{S(\bigoplus_{l \in I} P_l)} - i \circ j_l(1_{P_l}) \mid x_l, y_l \in P_l, l \in I\}$ ,  $\nu : S(\bigoplus_{l \in I} P_l) \rightarrow S(\bigoplus_{l \in I} P_l)/\overline{J}$  denotes the canonical projection and  $q_l = \nu \circ i \circ j_l$ , for all  $l \in I$ . Since the idea behind this construction is essentially the same as the one used in the case of associative algebras we will be brief. Consider  $Q$  to be a Poisson algebra and  $u_r : P_r \rightarrow Q$  a family of Poisson algebra maps. Composing the universal maps depicted in (3) it yields a unique Poisson algebra map  $u : S(\bigoplus_{l \in I} P_l)/\overline{J} \rightarrow Q$  such that  $u \circ q_r = u_r$ :

$$(3) \quad \begin{array}{ccccccc} P_r & \xrightarrow{j_r} & \bigoplus_{l \in I} P_l & \xrightarrow{i} & S(\bigoplus_{l \in I} P_l) & \xrightarrow{\nu} & S(\bigoplus_{l \in I} P_l) / \bar{J} \\ & \searrow u_r & \searrow \bar{u} & & \searrow \underline{u} & & \searrow u \\ & & & & & & Q \end{array}$$

Next in line are coequalizers. Let  $f, g : P \rightarrow Q$  be two Poisson algebra maps and consider  $\bar{I}$  to be the Poisson ideal generated by the set  $\{f(p) - g(p) \mid p \in P\}$ . Then  $(Q/\bar{I}, \pi)$  is the coequalizer of the morphisms  $(f, g)$  in  $k\text{-Poiss}$ , where  $\pi : Q \rightarrow Q/\bar{I}$  is the canonical projection; the details are straightforward and left to the reader.  $\square$

In order to have a complete picture on the category  $k\text{-Poiss}$  of Poisson algebras we record here the result concerning limits:

**Proposition 2.2.** *The category  $k\text{-Poiss}$  of Poisson algebras has arbitrary products and equalizers. In particular, the category  $k\text{-Poiss}$  of Poisson algebras is complete.*

*Proof.* It is straightforward to see that both products as well as equalizers can be constructed as simply the products and respectively the equalizers of the underlying vector spaces with the obvious commutative algebra and Lie algebra structures.  $\square$

As mentioned before, the colimits construction in the categories  $k\text{-BiAlgPoiss}$  and  $k\text{-HopfPoiss}$  rely heavily on the corresponding construction performed in the category  $k\text{-Poiss}$ .

**Theorem 2.3.** *The categories  $k\text{-BiAlgPoiss}$  and  $k\text{-HopfPoiss}$  of Poisson bialgebras and respectively Poisson Hopf algebras have arbitrary coproducts and coequalizers. In particular, the above categories are cocomplete.*

*Proof.* First we deal with products in the category of Poisson bialgebras. Consider a family of Poisson bialgebras  $(B_i, m_i, \eta_i, \Delta_i, \varepsilon_i, [-, -]_i)_{i \in I}$  and let  $(\coprod_{i \in I} B_i, (q_i)_{i \in I})$  be the coproduct of this family in the category  $k\text{-Poiss}$  as described in Proposition 2.1. It turns out that  $\coprod_{i \in I} B_i$  is actually a Poisson bialgebra with comultiplication and counit given by the unique Poisson algebra maps such that the following diagrams commute:

$$(4) \quad \begin{array}{ccc} B_l & \xrightarrow{q_l} & \coprod_{i \in I} B_i \\ & \searrow (q_l \otimes q_l) \circ \Delta_l & \downarrow \Delta \\ & & \coprod_{i \in I} B_i \otimes \coprod_{i \in I} B_i \end{array} \quad \begin{array}{ccc} B_l & \xrightarrow{q_l} & \coprod_{i \in I} B_i \\ & \searrow \varepsilon_l & \downarrow \varepsilon \\ & & k \end{array}$$

Proving that  $(\coprod_{i \in I} B_i, \Delta, \varepsilon)$  is a coalgebra goes essentially in the same vain as in the case of Hopf algebras; we refer the reader to [14] for more details.

Next we look at coequalizers. Let  $f, g : P \rightarrow Q$  be two Poisson bialgebra maps and consider  $\bar{I}$  to be the Poisson ideal of  $Q$  generated by the set  $\{f(p) - g(p) \mid p \in P\}$ . We prove that  $\bar{I}$  is also a coideal, i.e.  $\Delta(\bar{I}) \subseteq Q \otimes \bar{I} + \bar{I} \otimes Q$ . In fact, as  $\Delta$  is a Poisson algebra

morphism we only need to check that  $\Delta([q, r(f(p) - g(p))]) \subseteq Q \otimes \bar{I} + \bar{I} \otimes Q$ , for all  $q, r \in Q$  and  $p \in P$ . Indeed, using induction this would imply that  $\Delta([q_1, [q_2, \dots, [q_k, r(f(p) - g(p))]]]) \subseteq Q \otimes \bar{I} + \bar{I} \otimes Q$  for all  $k \in \mathbb{N}$ ,  $q_1, q_2, \dots, q_k, r \in Q$ ,  $p \in P$  and the conclusion follows. Now we prove our original claim:

$$\begin{aligned}
& \Delta([q, r(f(p) - g(p))]) = [\Delta(q), \Delta(r(f(p) - g(p)))]_{Q \otimes Q} \\
&= [q_{(1)} \otimes q_{(2)}, (r_{(1)} \otimes r_{(2)})(f(p_{(1)}) \otimes f(p_{(2)}) - g(p_{(1)}) \otimes g(p_{(2)}))]_{Q \otimes Q} \\
&= [q_{(1)} \otimes q_{(2)}, (r_{(1)} \otimes r_{(2)})(f(p_{(1)}) \otimes (f(p_{(2)}) - g(p_{(2)}))) - \\
&\quad (r_{(1)} \otimes r_{(2)})((f(p_{(1)}) - g(p_{(1)})) \otimes g(p_{(2)}))]_{Q \otimes Q} \\
&= [q_{(1)} \otimes q_{(2)}, r_{(1)}f(p_{(1)}) \otimes r_{(2)}(f(p_{(2)}) - g(p_{(2)}))]_{Q \otimes Q} - \\
&\quad [q_{(1)} \otimes q_{(2)}, r_{(1)}(f(p_{(1)}) - g(p_{(1)})) \otimes r_{(2)}g(p_{(2)})]_{Q \otimes Q} \\
&\stackrel{(1)}{=} \underline{q_{(1)}r_{(1)}f(p_{(1)}) \otimes [q_{(2)}, r_{(2)}(f(p_{(2)}) - g(p_{(2)}))]} + \\
&\quad \underline{[q_{(1)}, r_{(1)}f(p_{(1)})] \otimes q_{(2)}r_{(2)}(f(p_{(2)}) - g(p_{(2)}))} - \\
&\quad \underline{q_{(1)}r_{(1)}(f(p_{(1)}) - g(p_{(1)})) \otimes [q_{(2)}, r_{(2)}g(p_{(2)})]} - \\
&\quad \underline{[q_{(1)}, r_{(1)}(f(p_{(1)}) - g(p_{(1)}))] \otimes q_{(2)}r_{(2)}g(p_{(2)})}
\end{aligned}$$

and the last line is obviously in  $Q \otimes \bar{I} + \bar{I} \otimes Q$ , as the underlined terms are in  $\bar{I}$ . Therefore,  $Q/\bar{I}$  is a Poisson bialgebra in a canonical way and moreover  $(Q/\bar{I}, \pi)$  is the coequalizer of the morphisms  $(f, g)$  in  $k\text{-BiAlgPoiss}$ , where  $\pi : Q \rightarrow Q/\bar{I}$  is the canonical projection.

Consider now a family of Poisson Hopf algebras  $(H_i, m_i, \eta_i, \Delta_i, \varepsilon_i, S_i, [-, -]_i)_{i \in I}$  and let  $(H := \coprod_{i \in I} H_i, m, \eta, \Delta, \varepsilon, [-, -], (q_i)_{i \in I})$  be the previously constructed coproduct of the underlying Poisson bialgebras. Remark that  $S_i : H_i \rightarrow H_i^{\text{op}, \text{cop}}$  is a Poisson bialgebra map. Then, the universal property of the coproduct yields an unique Poisson bialgebra map  $S : H \rightarrow H^{\text{op}, \text{cop}}$  such that the following diagram commutes for all  $i \in I$ :

$$(5) \quad \begin{array}{ccc} H_i & \xrightarrow{q_i} & H \\ & \searrow q_i \circ S_i & \downarrow S \\ & & H^{\text{op}, \text{cop}} \end{array}$$

As  $S : H \rightarrow H^{\text{op}, \text{cop}}$  defined in (5) is a Poisson bialgebra map we only need to prove that  $S$  is indeed an antipode for  $H$ . This follows exactly as in the proof of [3, Theorem 2.2]. Finally, since  $k\text{-HopfPoiss}$  is a full subcategory of the category  $k\text{-BiAlgPoiss}$  it follows that  $(H := \coprod_{i \in I} H_i, \Delta, \varepsilon, S, [-, -], (q_i)_{i \in I})$  is also the coproduct in  $k\text{-HopfPoiss}$ .

Consider now  $f, g : P \rightarrow Q$  to be two Poisson Hopf algebra maps and, as before, let  $\bar{J}$  be the Poisson ideal of  $Q$  generated by the set  $\{f(p) - g(p) \mid p \in P\}$ . The computations performed in the case of Poisson bialgebras imply that  $\bar{J}$  is also a coideal. We are left to prove that  $\bar{J}$  is a Hopf ideal, i.e.  $S_Q(\bar{J}) \subseteq \bar{J}$ . Arguing as in the first part of the proof, we only need to show that  $S_Q([q, r(f(p) - g(p))]) \subseteq \bar{J}$  for all  $q, r \in Q$  and  $p \in P$ . Indeed,

we have:

$$\begin{aligned}
S_Q([q, r(f(p) - g(p))]) &= [S_Q(r(f(p) - g(p))), S_Q(q)] \\
&= [S_Q(f(p) - g(p)) S_Q(r), S_Q(q)] \\
&= [(f(S_P(p)) - g(S_P(p))) S_Q(r), S_Q(q)] \\
&= -[S_Q(q), S_Q(r)(f(S_P(p)) - g(S_P(p)))] \in \overline{\mathcal{J}}
\end{aligned}$$

and the proof is now finished.  $\square$

### 3. THE FREE POISSON HOPF ALGEBRA ON A COALGEBRA

In this section we introduce the main characters of this paper, namely the free Poisson Hopf algebras generated by coalgebras. The strategy we pursue is the following: first, we introduce the free Poisson bialgebra on a coalgebra (Theorem 3.1) and then we prove that there also exist a free Poisson Hopf algebra on every Poisson bialgebra (Theorem 3.2). Finally, by putting the two constructions together we arrive at the free Poisson Hopf algebra generated by a coalgebra (Theorem 3.3).

**Theorem 3.1.** *The forgetful functor  $F_1 : k\text{-BiAlgPois} \rightarrow k\text{-CoAlg}$  has a left adjoint, i.e. there exists a free Poisson bialgebra on every coalgebra.*

*Proof.* Let  $(C, \Delta_C, \varepsilon_C)$  be a coalgebra and consider  $(\mathcal{P}(C), \bar{i})$  to be the free Poisson algebra on the vector space  $C$ . By the universal property of the free Poisson algebra we obtain two Poisson algebra maps  $\overline{\Delta} : \mathcal{P}(C) \rightarrow \mathcal{P}(C) \otimes \mathcal{P}(C)$  and respectively  $\bar{\varepsilon} : \mathcal{P}(C) \rightarrow k$  such that the following two diagrams commute:

$$(6) \quad \begin{array}{ccc} C & \xrightarrow{\bar{i}} & \mathcal{P}(C) \\ & \searrow (\bar{i} \otimes \bar{i}) \circ \Delta_C & \downarrow \overline{\Delta} \\ & & \mathcal{P}(C) \otimes \mathcal{P}(C) \end{array} \quad \begin{array}{ccc} C & \xrightarrow{\bar{i}} & \mathcal{P}(C) \\ & \searrow \varepsilon_C & \downarrow \bar{\varepsilon} \\ & & k \end{array}$$

We start by proving that  $(\mathcal{P}(C), \overline{\Delta}, \bar{\varepsilon})$  is a coalgebra. Consider the Poisson algebra map  $(\overline{\Delta} \otimes Id) \circ \overline{\Delta} \circ \bar{i}$ . The universal property of the free Poisson algebra yields a unique Poisson algebra map  $\psi : \mathcal{P}(C) \rightarrow \mathcal{P}(C) \otimes \mathcal{P}(C) \otimes \mathcal{P}(C)$  such that the following diagram commutes:

$$(7) \quad \begin{array}{ccc} C & \xrightarrow{\bar{i}} & \mathcal{P}(C) \\ & \searrow (\overline{\Delta} \otimes Id) \circ \overline{\Delta} \circ \bar{i} & \downarrow \psi \\ & & \mathcal{P}(C) \otimes \mathcal{P}(C) \otimes \mathcal{P}(C) \end{array}$$

It is easy to see that the Poisson algebra map  $(\overline{\Delta} \otimes Id) \circ \overline{\Delta}$  makes the above diagram commute. Thus, using the uniqueness of  $\psi$ , in order to prove that  $(\overline{\Delta} \otimes Id) \circ \overline{\Delta} =$



$(Id \otimes \overline{\Delta}) \circ \overline{\Delta}$  it is enough to show that  $(\overline{\Delta} \otimes Id) \circ \overline{\Delta} \circ \bar{i} = (Id \otimes \overline{\Delta}) \circ \overline{\Delta} \circ \bar{i}$ . Indeed, concerning the last claim we have:

$$\begin{aligned}
(\overline{\Delta} \otimes Id) \circ \underline{\overline{\Delta} \circ \bar{i}} &\stackrel{(6)}{=} (\overline{\Delta} \otimes Id) \circ (\bar{i} \otimes \bar{i}) \circ \Delta_C \\
&= ((\overline{\Delta} \circ \bar{i}) \otimes \bar{i}) \circ \Delta_C \\
&\stackrel{(6)}{=} \left( ((\bar{i} \otimes \bar{i}) \circ \Delta_C) \otimes \bar{i} \right) \circ \Delta_C \\
&= (\bar{i} \otimes \bar{i} \otimes \bar{i}) \circ (Id \otimes \Delta_C) \circ \Delta_C \\
&= (\bar{i} \otimes \bar{i} \otimes \bar{i}) \circ (\Delta_C \otimes Id) \circ \Delta_C \\
&= \left( \bar{i} \otimes (\bar{i} \otimes \bar{i}) \circ \Delta_C \right) \circ \Delta_C \\
&\stackrel{(6)}{=} (\bar{i} \otimes (\overline{\Delta} \circ \bar{i})) \circ \Delta_C \\
&= (Id \otimes \overline{\Delta}) \circ (\bar{i} \otimes \bar{i}) \circ \Delta_C \\
&\stackrel{(6)}{=} (Id \otimes \overline{\Delta}) \circ \overline{\Delta} \circ \bar{i}
\end{aligned}$$

A similar argument proves that  $(Id \otimes \overline{\varepsilon}) \circ \overline{\Delta} = (\overline{\varepsilon} \otimes Id) \circ \overline{\Delta} = Id$  and thus  $\mathcal{P}(C)$  is in fact a Poisson bialgebra. The proof will be finished once we show that the pair  $(\mathcal{P}(C), \bar{i})$  provides a co-universal solution to the co-universal problem generated by the coalgebra  $C$  and the forgetful functor  $F_1 : k\text{-BiAlgPois} \rightarrow k\text{-CoAlg}$ . To this end, let  $H$  be a Poisson bialgebra and  $f : C \rightarrow H$  a coalgebra map. By the universal property of the free Poisson algebra, there exists a Poisson algebra map  $\bar{f} : \mathcal{P}(C) \rightarrow H$  such that the following diagram is commutative:

$$\begin{array}{ccc}
C & \xrightarrow{\bar{i}} & \mathcal{P}(C) \\
& \searrow f & \downarrow \bar{f} \\
& & H
\end{array}$$

We are left to prove that  $\bar{f}$  is also a coalgebra map. Since  $\Delta_H \circ \bar{f}$  is a Poisson algebra map, by the universal property of the free Poisson algebra, there exists a unique Poisson algebra map  $\xi : \mathcal{P}(C) \rightarrow H \otimes H$  such that the following diagram commutes:

$$\begin{array}{ccc}
C & \xrightarrow{\bar{i}} & \mathcal{P}(C) \\
& \searrow \Delta_H \circ \bar{f} \circ \bar{i} & \downarrow \xi \\
& & H \otimes H
\end{array}$$

Obviously,  $\Delta_H \circ \bar{f}$  makes the above diagram commutative. By the same argument as before, in order to prove that  $(\bar{f} \otimes \bar{f}) \circ \overline{\Delta} = \Delta_H \circ \bar{f}$ , it will be enough to show that



$(\bar{f} \otimes \bar{f}) \circ \bar{\Delta} \circ \bar{i} = \Delta_H \circ \bar{f} \circ \bar{i}$ . As for the last claim, we have:

$$\begin{aligned}
 (\bar{f} \otimes \bar{f}) \circ \bar{\Delta} \circ \bar{i} &\stackrel{(6)}{=} (\bar{f} \otimes \bar{f}) \circ (\bar{i} \otimes \bar{i}) \circ \Delta_C \\
 &= ((\bar{f} \circ \bar{i}) \otimes (\bar{f} \circ \bar{i})) \circ \Delta_C \\
 &= (f \otimes f) \circ \Delta_C \\
 &\stackrel{f \text{ coalgebra map}}{=} \Delta_H \circ f = \Delta_H \circ \bar{f} \circ \bar{i}
 \end{aligned}$$

Consider now the Poisson algebra map  $\varepsilon_H \circ \bar{f} \circ \bar{i}$ . Again by the universal property of the free Poisson algebra, there exists a unique Poisson algebra map  $\chi : \mathcal{P}(C) \rightarrow k$  such that the following diagram is commutative:

$$\begin{array}{ccc}
 C & \xrightarrow{\bar{i}} & \mathcal{P}(C) \\
 & \searrow \varepsilon_H \circ \bar{f} \circ \bar{i} & \downarrow \chi \\
 & & k
 \end{array}$$

Using the same argument as before, in order to prove that  $\varepsilon_H \circ \bar{f} = \bar{\varepsilon}$  we only need to show that  $\varepsilon_H \circ \bar{f} \circ \bar{i} = \bar{\varepsilon} \circ \bar{i}$ . The latter statement holds true by the following straightforward computation:

$$\varepsilon_H \circ \bar{f} \circ \bar{i} = \varepsilon_H \circ f = \varepsilon_C = \bar{\varepsilon} \circ \bar{i}$$

This finishes the proof.  $\square$

In the proof of our next theorem we make use of some statements already proven in [14, Theorem 2.6.3]. Since the proof there is very detailed and freely available, we will refer to it and leave out those computations.

**Theorem 3.2.** *The forgetful functor  $F_2 : k\text{-PoisHopf} \rightarrow k\text{-BiAlgPois}$  has a left adjoint, i.e. there exists a free Poisson Hopf algebra on every Poisson bialgebra.*

*Proof.* Let  $B$  be a Poisson bialgebra. We aim to construct a co-universal solution to the co-universal problem generated by  $B$  and the forgetful functor  $F_2 : k\text{-PoisHopf} \rightarrow k\text{-BiAlgPois}$ . To this end, consider  $(C = \coprod_{n \in \mathbb{N}} B_n, (q_n)_{n \in \mathbb{N}})$  to be the coproduct in  $k\text{-BiAlgPois}$  of the Poisson bialgebras  $B_n$ ,  $n \geq 0$ , where  $B_n = B$  for  $n$  even and  $B_n = B^{\text{op}, \text{cop}}$  for  $n$  odd. The universality of the coproduct yields a unique Poisson bialgebra map  $S' : C \rightarrow C^{\text{op}, \text{cop}}$  which makes the following diagram commutative:

$$\begin{array}{ccc}
 B_i & \xrightarrow{q_i} & C \\
 & \searrow Id & \downarrow S' \\
 & & B_{i+1}^{\text{op}, \text{cop}} \\
 & & \searrow q_{i+1} \\
 & & C^{\text{op}, \text{cop}}
 \end{array}$$

Now consider  $\mathcal{I}$  to be the Poisson ideal of  $C$  generated by the set  $I = \{(S' * Id - u_C \circ \varepsilon_C)(q_n(x)), (Id * S' - u_C \circ \varepsilon_C)(q_n(x)) \mid x \in B_n, n \in \mathbb{N}\}$ . We will prove that  $\mathcal{I}$

is also a Hopf ideal, i.e.  $\varepsilon_C(\mathcal{I}) = 0$ ,  $\Delta_C(\mathcal{I}) \subseteq C \otimes \mathcal{I} + \mathcal{I} \otimes C$  and  $S'(\mathcal{I}) \subseteq \mathcal{I}$ . As noted in the proof of Theorem 2.3, it is enough to check this on the elements of the form  $[a, b(S'(q_n(x)_{(1)})q_n(x)_{(2)} - u_C \circ \varepsilon_C(q_n(x)))]$  where  $a, b \in C$  and  $x \in B_n$ ,  $n \in \mathbb{N}$ . We know from [14, Theorem 2.6.3] that  $\Delta_C(S'(q_n(x)_{(1)})q_n(x)_{(2)} - u_C \circ \varepsilon_C(q_n(x))) \in C \otimes CI + CI \otimes C$ , where  $CI$  denotes the two-sided ideal generated by  $I$ . Since the inclusion  $CI \subset \mathcal{I}$  holds true, we will denote  $\Delta_C(S'(q_n(x)_{(1)})q_n(x)_{(2)} - u_C \circ \varepsilon_C(q_n(x))) = c \otimes \iota + \bar{\iota} \otimes \bar{c}$  (note that in order to be consistent with our notations, we suppressed the summation sign in the right hand side) with  $c, \bar{c} \in C$  and  $\iota, \bar{\iota} \in \mathcal{I}$ . Then, we have:

$$\begin{aligned}
& \Delta_C \left( [a, b(S'(q_n(x)_{(1)})q_n(x)_{(2)} - u \circ \varepsilon(q_n(x)))]_C \right) = \\
&= [\Delta(a), \Delta(b)\Delta(S'(q_n(x)_{(1)})q_n(x)_{(2)} - u \circ \varepsilon(q_n(x)))]_{C \otimes C} \\
&= [a_{(1)} \otimes a_{(2)}, (b_{(1)} \otimes b_{(2)})(c \otimes \iota + \bar{\iota} \otimes \bar{c})]_{C \otimes C} \\
&\stackrel{(1)}{=} [a_{(1)} \otimes a_{(2)}, b_{(1)}c \otimes b_{(2)}\iota]_{C \otimes C} + [a_{(1)} \otimes a_{(2)}, b_{(1)}\bar{\iota} \otimes b_{(2)}\bar{c}]_{C \otimes C} \\
&\stackrel{(1)}{=} a_{(1)}b_{(1)}c \otimes \underline{[a_{(2)}, b_{(2)}\iota]_C} + [a_{(1)}, b_{(1)}c]_C \otimes \underline{a_{(2)}b_{(2)}\iota} + \underline{a_{(1)}b_{(1)}\bar{\iota}} \otimes [a_{(2)}, b_{(2)}\bar{c}]_C \\
&\quad + \underline{[a_{(1)}, b_{(1)}\bar{\iota}]_C} \otimes a_{(2)}b_{(2)}\bar{c}
\end{aligned}$$

and the conclusion follows since the underlined terms belong to  $\mathcal{I}$ . We are left to show that  $S'(\mathcal{I}) \subseteq \mathcal{I}$ . Recall from the proof of [14, Theorem 2.6.3] that  $S'(S'(q_n(x)_{(1)})q_n(x)_{(2)} - u_C \circ \varepsilon_C(q_n(x))) = q_{n+1}(x)_{(1)}S'(q_{n+1}(x)_{(2)}) - u_C \circ \varepsilon_C(q_{n+1}(x)) \in I$  for all  $x \in B_n$ ,  $n \in \mathbb{N}$ . Therefore, we have:

$$\begin{aligned}
& S' \left( [a, b(S'(q_n(x)_{(1)})q_n(x)_{(2)} - u \circ \varepsilon(q_n(x)))]_C \right) = \\
&= [S'(S'(q_n(x)_{(1)})q_n(x)_{(2)} - u \circ \varepsilon(q_n(x)))S'(b), S'(a)]_C \\
&= [\underline{(q_{n+1}(x)_{(1)}S'(q_{n+1}(x)_{(2)}) - u \circ \varepsilon(q_{n+1}(x)))} S'q_{n+1}(x)_{(1)}S'(b), S'(a)]_C
\end{aligned}$$

and the last line is clearly in  $\mathcal{I}$  since the underlined term belongs to  $I$ .

We are now ready to construct a co-universal solution to the co-universal problem generated by the Poisson bialgebra  $B$  and the functor  $F_2 : k\text{-PoisHopf} \rightarrow k\text{-BiAlgPois}$ . To this end, consider the Poisson algebra  $H(C) := C/\mathcal{I}$  and  $\pi : C \rightarrow C/\mathcal{I}$  the canonical projection.  $H(C)$  can be made into a Poisson Hopf algebra with the coalgebra structure  $(\bar{\Delta}, \bar{\varepsilon})$  and antipode  $\bar{S}$  given by the unique Poisson algebra maps which make the following diagrams commute:

$$\begin{array}{ccc}
C \xrightarrow{\pi} H(C) & C \xrightarrow{\pi} H(C) & C \xrightarrow{\pi} H(C) \\
\searrow \varepsilon & \searrow (\pi \otimes \pi) \circ \Delta & \searrow \pi \circ S' \\
& \downarrow \bar{\varepsilon} & \downarrow \bar{\Delta} & \downarrow \bar{S} \\
& k & H(C) \otimes H(C) & H(C)^{\text{cop}}
\end{array}$$

By arguments similar to those used in the proof of [14, Theorem 2.6.3] it can be easily seen that the object constructed above indeed provides a co-universal solution to the co-universal problem generated by the Poisson bialgebra  $B$  and the functor  $F_2 : k\text{-PoissHopf} \rightarrow k\text{-BiAlgPoiss}$ .  $\square$

We can now put the last two results together:

**Theorem 3.3.** *The forgetful functor  $F : k\text{-PoissHopf} \rightarrow k\text{-CoAlg}$  has a left adjoint, i.e. there exists a free Poisson Hopf algebra on every coalgebra.*

*Proof.* It follows by composing the two left adjoint functors constructed in Theorem 3.1 and Theorem 3.2.  $\square$

**Corollary 3.4.** *The categories  $k\text{-PoissHopf}$  and  $k\text{-BiAlgPoiss}$  of Poisson Hopf algebras and respectively Poisson bialgebras have generators.*

*Proof.* We denote by  $\overline{F}_1 : k\text{-CoAlg} \rightarrow k\text{-BiAlgPoiss}$  and  $\overline{F} : k\text{-CoAlg} \rightarrow k\text{-PoissHopf}$  the left adjoint functors of  $F_1 : k\text{-BiAlgPoiss} \rightarrow k\text{-CoAlg}$  and respectively  $F : k\text{-PoissHopf} \rightarrow k\text{-CoAlg}$ . By [16, Theorem 17], the category  $k\text{-CoAlg}$  of coalgebras has a generator  $G$ . Recall that  $G = \coprod_{x \in I} C_x$ , where  $I$  is the set of isomorphism classes of finite dimensional coalgebras over  $k$  while  $C_x$  is a coalgebra in the isomorphism class of  $x \in I$ . Now since both forgetful functors  $F_1$  and  $F$  are faithful it follows that  $\overline{F}_1(G)$  and  $\overline{F}(G)$  are generators in the categories of Poisson bialgebras and respectively Poisson Hopf algebras.  $\square$

#### SOME COMMENTS AND OPEN PROBLEMS

The main results of this paper are Theorem 2.3 and Theorem 3.3 which prove the cocompleteness of the categories  $k\text{-BiAlgPoiss}$  and  $k\text{-PoissHopf}$ , and respectively the existence of a free Poisson Hopf algebra on every coalgebra. In order to complete the categorical picture drawn by the above results, it is natural to ask the following:

**Question 1:** Are the categories  $k\text{-BiAlgPoiss}$  and respectively  $k\text{-PoissHopf}$  complete (i.e. do they have arbitrary products and equalizers)?

It is worth pointing out that, if limits in  $k\text{-BiAlgPoiss}$  and respectively  $k\text{-PoissHopf}$  do exist, then Theorem 3.3 tells us that they should be constructed as simply the limits of the underlying coalgebras.

**Question 2:** Does there exist a cofree Poisson Hopf algebra on every Poisson algebra, respectively on every Poisson bialgebra? (i.e. do the forgetful functors  $U : k\text{-PoissHopf} \rightarrow k\text{-Poiss}$  and respectively  $\overline{U} : k\text{-PoissHopf} \rightarrow k\text{-BiAlgPoiss}$  have right adjoints?)

Another important issue, also related to the existence of cofree objects in the above mentioned categories, is the injectivity (resp. surjectivity) of monomorphisms (resp. epimorphisms). It was proven in [4] that monomorphisms (resp. epimorphisms) in the category  $k\text{-HopfAlg}$  of Hopf algebras are not necessarily injective (reps. surjective) maps. This was achieved by noticing that the antipode of any Hopf algebra is both a monomorphism and an epimorphism in the category  $k\text{-HopfAlg}$  of Hopf algebras together with

the well known fact that there exist Hopf algebras with non-injective, respectively non-surjective antipode. Although the result in [4] holds true in the category  $k\text{-PoissHopf}$  of Poisson Hopf algebras as well, it has no implications on the injectivity (resp. surjectivity) of monomorphisms (resp. epimorphisms) due to the fact that all Poisson Hopf algebras have bijective antipodes as a consequence of being commutative Hopf algebras.

As noted above, we do not know whether epimorphisms in the aforementioned categories are surjective maps. This information is of particular interest mainly in connection to the Special Adjoint Functor Theorem (see [12, Corollary p. 130]) which, under some additional assumptions, provides necessary and sufficient conditions for a functor to be a left (respectively right) adjoint. Having a positive answer to the question of whether epimorphisms in  $k\text{-PoissHopf}$  are surjective maps would also settle the co-wellpoweredness problem into positive and therefore all the assumptions required for applying the Special Adjoint Functor Theorem would be fulfilled. Thus, the forgetful functors  $U$  and  $\bar{U}$  would have right adjoints and therefore we would be able to construct the cofree Poisson Hopf algebra generated by a Poisson algebra, respectively a Poisson bialgebra. On the other hand, a negative answer to the question of whether epimorphisms are surjective maps in  $k\text{-PoissHopf}$  it has no implications on the co-wellpoweredness of the category  $k\text{-PoissHopf}$ : for instance, the category  $k\text{-HopfAlg}$  of Hopf algebras is co-wellpowered although epimorphisms are not necessarily surjective maps (see [3, 4, 17]).

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